Week 8: Homotopy Pushouts I

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1 Introduction

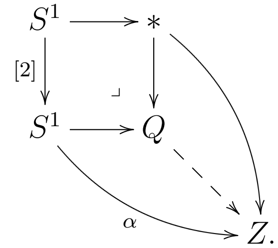
In the very first exercise sheet we constructed some basic limits and colimits in the homotopy category, namely products and coproducts. Nevertheless, we went on to comment that we cannot expect the homotopy category to have more complicated limits and colimits.

Thus we begin this week's exercise sheet by demonstrating that the pointed homotopy category does not have pushouts. This motivates the construction of so-called *homotopy pushouts*, whose study shall be taken up in the remainder of the exercise sheet. These objects are constructed not in the homotopy category, but rather in Top_* itself. It is unfair to think of them merely as 'replacements' for pushouts in $hTop_*$, since they actually do a lot more. The story is analogous to that of the mapping cone construction, which although gives a homotopy-invariant way to form quotient spaces, ends up leading to interesting behaviour such as infinite cofiber sequences.

We work in the pointed category throughout. Please complete all exercises. There are five in total.

2 The Homotopy Category Does Not Have Pushouts

We work in $hTop_*$ although the story is the same without basepoints. To see that pushouts do not exist we construct an absurdity. Let us assume that the square in the following diagram is a pushout in $hTop_*$



(2.1)

Here[2] is the homotopy class of the degree 2 map $2 : S^1 \to S^1$. If the pushout space Q exists, then the universal property it must satisfy for a given space Z reads as follows

$$[Q, Z] \cong \{\alpha \in \pi_1 Z \mid 2 \cdot \alpha = 0\} \cong Hom(\mathbb{Z}_2, \pi_1 Z). \tag{2.2}$$

Now consider the fibration sequence

$$S^1 \to SO_3 \to S^2. \tag{2.3}$$

Applying $[Q, -]$ gives to this fibration sequence gives rise to the exact sequence

$$[Q, S^1] \to [Q, SO_3] \to [Q, S^2] \tag{2.4}$$

By (2.2) we have $[Q, S^1] \cong Hom(\mathbb{Z}_2, \mathbb{Z}) = 0$ and $[Q, S^2] = 0$, while $[Q, SO_3] \cong [Q, \mathbb{R}P^3] \cong Hom(\mathbb{Z}_2, \mathbb{Z}_2)$. This means that the sequence (2.4) is

$$0 \to \mathbb{Z}_2 \to 0 \tag{2.5}$$

and this is clearly not exact. Hence we rule out the existence any space Q which would satisfy the universal property of the pushout.

On the other hand, while we may not be able to form pushouts, we can form form *weak* pushouts. As soon as we drop the requirement that the induced homotopy class be unique we get some insight on how to proceed.

Definition 1 *Let $\mathcal{C}$ be a category. A commutative diagram*

$$\begin{CD} W @>{g}>> Y \\ @V{f}VV @VV{k}V \\ X @>{h}>> P \end{CD} \tag{2.6}$$

of morphisms is $\mathcal{C}$ is said to be a ***weak pushout*** *if whenever given morphisms $X \stackrel{\alpha}{\to} Q \stackrel{\beta}{\leftarrow} Y$ with $\alpha f = \beta g$ there exists a morphism $\gamma : P \to Q$ satisfying i) $\gamma h = \alpha$ and ii) $\gamma k = \beta$. The square* (2.6) *is said to be a* ***pushout*** *if it is a weak pushout and if for each pair of morphisms $X \stackrel{\alpha}{\to} Q \stackrel{\beta}{\leftarrow} Y$ with $\alpha f = \beta g$, the induced morphism $\gamma : P \to Q$ is unique.* □

We have already seen examples of weak pushouts. Recall that if $A \hookrightarrow X$ is a cofibration, then

$$\begin{CD} A @>>> A \wedge I_+ \\ @VVV @VVV \\ X @>>> X \wedge I_+ \end{CD} \tag{2.7}$$

is a weak pushout in Top_*.

As another example, let us return to (2.1). We claim that if $Q = \mathbb{R}P^2$, then this square does have the weak pushout property in the homotopy category. In fact we have already seen how to turn a null composite $2 \cdot \alpha = \alpha \circ 2 \simeq$ into a map $\mathbb{R}P^2 \to Z$ out of the mapping cone of $2 : S^1 \to S^1$. Namely we choose representatives for each homotopy class and make a particualr choice of null homotopy $F : 2 \cdot \alpha \simeq *$. From these choices we get the extension $\underline{\alpha}_F$ indicated in the right-hand diagram below

$$\begin{array}{ccc} S^1 & \longrightarrow & * \\ {\scriptstyle 2}\downarrow & \overset{F}{\Rightarrow} & \downarrow \\ S^1 & \xrightarrow{\alpha} & Z \end{array} \qquad\qquad \begin{array}{cccc} S^1 & \longrightarrow & D^2 & \\ {\scriptstyle 2}\downarrow & \lrcorner & \downarrow & \searrow^{F} \\ S^1 & \longrightarrow & \mathbb{R}P^2 & \\ & \searrow_{\alpha} & & \overset{\underline{\alpha}_F}{\dashrightarrow} Z. \end{array} \tag{2.8}$$

We stress that neither of these two diagrams belongs to $hTop_*$. The right-hand diagram belongs to Top_*, and the left-hand diagram lives somewhere between these two categories. However, when we project the right-hand diagram into $hTop_*$, we see that we are able to solve the pushout problem (2.1) in a weak sense.

Exercise 2.1 Show that if

$$X \overset{f}{\to} Y \overset{g}{\to} Z \tag{2.9}$$

is a homotopy cofiber sequence, then

$$\begin{array}{ccc} X & \longrightarrow & * \\ {\scriptstyle [f]}\downarrow & & \downarrow \\ Y & \xrightarrow{[g]} & Z \end{array} \tag{2.10}$$

is a weak pushout in the homotopy category. $\square$

Is is possible to that the weak pushout of an arbitrary span $X \leftarrow W \to Y$ exists in $hTop_*$? The answer is yes, but as these things go we will have more control over the situation by setting up our machinery inside Top_*. Our solution will be to construct *homotopy pushouts*. We stress that these are not pushouts in any categorical sense, and although we have used the idea of weak pushouts to motivate our work, the outcome will be something much more powerful.

3 Homotopy Pushouts

As promised we work in the category of pointed spaces. There are versions of all the constructions and statements of this section also in the unpointed category, and to be able to transition between the two categories we can often get away with assuming that all spaces are well-pointed.

Definition 2 *Let maps*

$$X \xleftarrow{f} W \xrightarrow{g} Y \tag{3.1}$$

be given. The ***double mapping cylinder*** *of f, g is the quotient space*

$$M(f,g) = \frac{X \vee (W \wedge I_+) \vee Y}{f(w) \sim (w,0) \quad (w,1) \sim g(w)}. \tag{3.2}$$

We denote the obvious inclusion maps by

$$i_X : X \to M(f,g), \qquad i_Y : Y \to M(f,g). \tag{3.3}$$

There is a canonical homotopy $\psi = \psi_{f,g} : i_X f \simeq i_Y g$ given by

$$\psi_t(w) = (w,t), \qquad w \in W, t \in I. \tag{3.4}$$

This homotopy fits in the following square

$$\begin{array}{ccc} W & \xrightarrow{g} & Y \\ {\scriptstyle f}\downarrow & \overset{\psi}{\Rightarrow} & \downarrow{\scriptstyle i_Y} \\ X & \xrightarrow{i_X} & M(f,g) \end{array} \tag{3.5}$$

which we call the ***standard homotopy pushout*** *of f, g.* $\square$

Example 3.1 *The standard homotopy pushout of $X \xleftarrow{f} W \to *$ is the mapping cone C_f. In particular the standard homotopy pushout of $* \leftarrow W \to *$ is the suspension ΣW.* $\square$

Suppose given a square with homotopy

$$\begin{array}{ccc} W & \xrightarrow{g} & Y \\ {\scriptstyle f}\downarrow & \overset{F}{\Rightarrow} & \downarrow{\scriptstyle k} \\ X & \xrightarrow{h} & Z. \end{array} \tag{3.6}$$

Then we get a comparison map

$$\theta_F : M(f,g) \to Z \tag{3.7}$$

by setting

$$\begin{aligned} \theta_F(x) &= h(x) \\ \theta_F(w,t) &= F(x,t) \\ \theta_F(y) &= k(y). \end{aligned} \tag{3.8}$$

The map θ_F allows us to measure the deviation of the square (3.6) (with homotopy!) from being the standard homotopy pushout.

Definition 3 *A homotopy commutative square*

$$\begin{array}{ccc} W & \xrightarrow{g} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle k} \\ X & \xrightarrow{h} & Z. \end{array} \tag{3.9}$$

is said to be a ***homotopy pushout*** *if there exists a homotopy $F : hf \simeq kg$ for which the induced map*

$$\theta_F : M(f, g) \to Z \tag{3.10}$$

is a homotopy equivalence. □

We stress that for (3.9) to be a homotopy pushout it is not sufficient only that Z is homotopy equivalent to the double mapping cylinder $M(f, g)$. It is important that the homotopy equivalence be induced in a particular way.

Exercise 3.1 Show that the homotopy commutative diagram

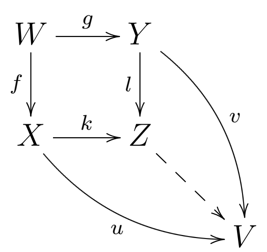
$$\begin{array}{ccc} X & \longrightarrow & * \\ {\scriptstyle f}\downarrow & & \downarrow \\ X & \xrightarrow{g} & Z \end{array} \tag{3.11}$$

is a homotopy pushout if and only if

$$X \xrightarrow{f} Y \xrightarrow{g} Z \tag{3.12}$$

is a homotopy cofiber sequence. □

Proposition 3.1 *Assume that the solid part of the following diagram commutes up to homotopy*



(3.13)

and that the square is a homotopy pushout. Then there is a map $m : Z \to K$ such that $mk \simeq u$ and $ml \simeq v$.

Proof We have already produced the required map when the square is the standard homotopy pushout of f, g, and the general case follows easily. ▮

Corollary 3.2 *The image of a homotopy pushout square in the homotopy category is a weak pushout.* ▮

We stress that there is much more information in the homotopy pushout square than in its image in the homotopy category. This is exactly the information of the homotopies, the presence of which allows for a level of control over the granted maps.

Next we will study when we should consider two different homotopy pushouts to be equivalent. Assume given a diagram of maps and homotopies of the form

$$\begin{array}{ccccc} X & \xleftarrow{f} & W & \xrightarrow{g} & Y \\ \downarrow{\scriptstyle\alpha} & \overset{F}{\Rightarrow} & \downarrow{\scriptstyle\gamma} & \overset{G}{\Rightarrow} & \downarrow{\scriptstyle\beta} \\ X' & \xleftarrow[f']{} & W' & \xrightarrow[g']{} & Y'. \end{array} \tag{3.14}$$

We use this data to define a map

$$\theta = \theta(F, G) : M(f, g) \to M(f', g') \tag{3.15}$$

between the homotopy pushouts of the rows by setting

$$\begin{aligned} \theta(x) &= \alpha(x) \\ \theta(w, t) &= \begin{cases} F(w, 3t) & 0 \le t \le \frac{1}{3} \\ (\gamma(w), 3t - 1) & \frac{1}{3} \le t \le \frac{2}{3} \\ G(w, 3t - 2) & \frac{2}{3} \le t \le 1 \end{cases} \\ \theta(y) &= \beta(x). \end{aligned} \tag{3.16}$$

Theorem 3.3 *With the notation of the last paragraph, if each of the maps α, β, γ is a homotopy equivalence, then so is the map $\theta : M(f, g) \to M(f', g')$.* ∎

We'll take this for granted. A proof can be found on pg. 314 of Arkowitz's book. For the most part it isn't difficult when the correct inverse has been constructed. The technical part of the proof lies in finding the correct homotopies to induce the comparison maps.

The theorem shows that the double mapping cylinder can be constructed on the level of the homotopy category. Taking α, β, γ to be identities we get the following.

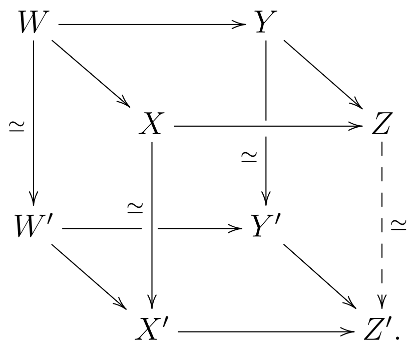
Corollary 3.4 *If $f \simeq f' : W \to X$ and $g \simeq g' : W \to Y$, then $M(f, g) \simeq M(f', g')$.* ∎

The theorem also shows that the homotopy pushout rectifies the failure of the categorical pushout to preserve pointwise equivalences of diagrams. Consider

$$\begin{array}{ccccc} D^n & \longleftarrow & S^{n-1} & \longrightarrow & D^n \\ \downarrow & & \| & & \downarrow \\ * & \longleftarrow & S^{n-1} & \longrightarrow & *. \end{array} \tag{3.17}$$

Each vertical arrow is a homotopy equivalence, but taking pushouts of the rows generates the map $S^n \to *$, which is not a homotopy equivalence. On the other hand, taking homotopy pushouts of the rows we induce a homotopy equivalence $S^n \xrightarrow{\simeq} S^n$ (although it need not be the identity).

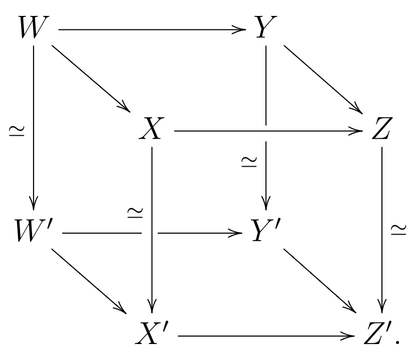
There is a clear extension of Theorem 3.3 to the case of arbitrary homotopy pushouts. We interpret this as saying that if the solid part of the following homotopy commutative diagram is given, if both the top and bottom squares are homotopy pushouts, and if the three vertical arrows are homotopy equivalences, then the dotted arrow can be filled in with a homotopy equivalence so as to make the entire diagram homotopy commute



(3.18)

Thus the homotopy pushout is essentially unique. In fact there is a sort of converse to this observation.

Theorem 3.5 *Assume given a homotopy commutative cube*



(3.19)

in which each vertical arrow is a homotopy equivalence. Then the top square is a homotopy pushout if and only if the bottom square is a homotopy pushout. ∎

The proof of this is more technical than difficult. The non-trivial point lies again in constructing suitable homotopies. Specifically, given homotopies F, K, L for the top, left and back squares of the cube we must find a homotopy F' for its bottom square so that the following diagram of comparison maps homotopy commutes

$$\begin{array}{ccc} M(f,g) & \xrightarrow{\theta_F} & D \\ {\scriptstyle \theta(K,L)}\downarrow \simeq & & \simeq \downarrow {\scriptstyle \delta} \\ M(f',g') & \xrightarrow{\theta_{F'}} & D. \end{array} \tag{3.20}$$

It is indeed possible to choose such an F', but its construction requires some work. In any case, once the homotopy commutativity of (3.20) is established, the proof of 3.5 follows easily from Theorem 3.3.

The theorems in this section will be needed for the exercises in the next two sections. We'll take them for granted for the moment, but we reassure the reader that the results will be covered in dual form in the lectures when I discuss *homotopy pullbacks*.

4 Recognising Homotopy Pushouts

Proposition 4.1 *Let*

$$\begin{array}{ccc} A & \longrightarrow & C \\ {\scriptstyle f}\big\downarrow{\scriptstyle \simeq} & & \big\downarrow{\scriptstyle k} \\ B & \longrightarrow & D \end{array} \tag{4.1}$$

commute up to homotopy. Assume that the map f is a homotopy equivalence. Then the square is a homotopy pushout if and only of the map $k : C \to D$ is a homotopy equivalence.

Note that the converse statement fails in general.

Exercise 4.1 Prove Proposition 4.1. □

Proposition 4.2 *Assume that the following strictly commutative square*

$$\begin{array}{ccc} W & \xrightarrow{g} & Y \\ {\scriptstyle f}\big\downarrow & \lrcorner & \big\downarrow \\ X & \longrightarrow & Z \end{array} \tag{4.2}$$

is a pushout. If either f or g is a cofibration, then the square is a homotopy pushout.

You'll be asked to prove Proposition 4.2 since it is quite important. The trick is to introduce the mapping cylinders M_f, M_g and notice that the double mapping cylinder $M(f,g)$ is homeomorphic to the categorical pushout of any one of the following spans

$$M_f \overset{j_f}{\longleftarrow} W \overset{j_g}{\longrightarrow} M_g, \qquad M_f \overset{j_f}{\longleftarrow} W \overset{g}{\to} Y, \qquad X \overset{f}{\leftarrow} W \overset{j_g}{\longrightarrow} M_g. \tag{4.3}$$

In fact this is always true, regardless of any special properties that the maps f, g might have.

Exercise 4.2 It suffices to prove Proposition 4.2 under the assumption that f is a cofibration. Use the strictly commutative diagram

$$\begin{array}{ccccc} M_f & \overset{j_f}{\longleftarrow} & W & \overset{g}{\longrightarrow} & Y \\ {\scriptstyle r_f}\big\downarrow{\scriptstyle \simeq} & & \Big\| & & \Big\| \\ X & \underset{f}{\longleftarrow} & W & \underset{g}{\longrightarrow} & Y. \end{array} \tag{4.4}$$

to induce a map $\theta : M(f,g) \to D$ between the pushouts of the rows. Use what you know about cofiber homotopy equivalences to show that θ is a homotopy equivalence and so prove 4.2. □

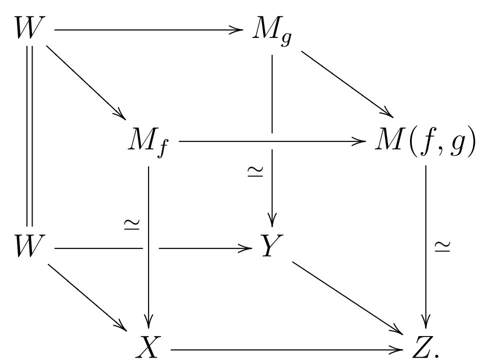
The observations proceeding this exercise are important enough to deserve a statement of their own.

Proposition 4.3 *Any homotopy pushout square is pointwise equivalent to strictly commutatitve square which is both a pushout and a homotopy pushout.* ∎

To explain what we have in mind let

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle f}\downarrow & & \downarrow \\ B & \longrightarrow & D \end{array} \tag{4.5}$$

be a given homotopy pushout. Then by converting the maps f, g into cofibrations be build a homotopy commutative cube of the form



(4.6)

The top face of the cube is both a pushout and a homotopy pushout, and moreover the left and back faces of the cube commute strictly.

5 Comparing Cofibers

Proposition 5.1 *Suppose that the homotopy commutative diagram*

$$\begin{array}{ccc} W & \xrightarrow{g} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle k} \\ X & \xrightarrow{h} & Z \end{array} \tag{5.1}$$

is a homotopy pushout. Then $C_f \simeq C_k$ and $C_g \simeq C_h$.

Exercise 5.1 Time to prove 5.1! By symmetry it will suffice to prove that $C_f \simeq C_k$. Replace (5.1) with the pointwise equivalent pushout diagram

$$\begin{array}{ccc} W & \xrightarrow{j_g} & M_g \\ {\scriptstyle j_f}\downarrow & \lrcorner & \downarrow{\scriptstyle \widehat{k}} \\ M_f & \longrightarrow & M(f,g) \end{array} \tag{5.2}$$

and construct homotopy equivalences

$$C_{j_f} \xrightarrow{\simeq} C_f, \qquad C_{\hat{k}} \xrightarrow{\simeq} C_k. \tag{5.3}$$

Use these to complete the proof. (Hint: Remember exercise 3.1?). □